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Electrohydrodynamic stability of a highly viscous jet

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When a pendant drop of weakly conducting fluid is raised to a high electric potential, it frequently adopts the shape of a Taylor cone from whose apex a thin, charged jet is emitted. Such a jet can display surprising longevity, but eventually breaks up into fine droplets, a fact utilized in electro-spraying devices. This paper examines the linear stability of an incompressible cylindrical jet carrying surface charge q in a tangential electric field E, for various values of the permittivity ratio λ and the finite rate of charge relaxation, τ. The viscosity is assumed to be large. It is shown that all axisymmetric temporal modes can be stabilized for suitable values of (q, E), but sinuous modes with logarithmically large wavelengths are unstable. If these very long waves are excluded, the jet can sometimes be completely stabilized. It is also shown that an uncharged jet with low permittivity is unstable to sinuous waves for large E, contrary to previous belief.

1. Introduction

The influence of electric fields on the stability behaviour of free liquid jets has attracted interest for over a century. The heyday of electrohydrodynamics (EHD) was the late 1960s (Melcher & Taylor 1969), but recently electro-spraying technology has led to a resurgence of interest in EHD phenomena. When a liquid drop hanging from an orifice feels a strong electric field, its surface deforms in response to electrical stresses, frequently forming a Taylor cone (Taylor 1964). If the field is increased further, the cone loses stability, emitting a thin jet from its apex. Under some conditions this configuration, known as the EHD cone-jet, is essentially steady, although the emitted jet eventually breaks up into small drops. The diameter of these drops is much smaller than that of the orifice. A number of industrial processes make use of this phenomenon to generate a uniform fine spray, charged to within a high proportion of the Rayleigh limit (Rayleigh 1882).

Experimental studies have identified a variety of parameter ranges in which cone-jets exist (see for example Zeleny 1915; Taylor 1969; Hayati, Bailey & Tadros 1987*a,b*; Cloupeau & Prunet-Foch 1989; Fernández de la Mora *et al.* 1990). A number of physical processes co-exist, each with its own timescale. In particular, the timescale for viscous diffusion across the jet can be large or small compared with, say, the timescale for capillary action. The Reynolds number appropriate to a surface perturbation can thus be high or low.

In this paper we concentrate on the question of the stability at low Reynolds number of the emitted jet. Previous studies have considered either a charged jet in the absence of a tangential electric field (Bassett 1894; Taylor 1969; Saville 1971*b*), or an uncharged jet in a uniform tangential field (Saville 1970, 1971*a*). Some of these works have assumed a perfect conductor, so that charge relaxation can be considered instantaneous. But Saville (1971*a*) demonstrated that the charge relaxation instabilities of Melcher & Schwarz (1968) can occur for imperfect conductors.

When the jet is produced by means of the cone-jet process, both surface charge and tangential field are present, although not in amounts which are easy to predetermine. Furthermore, the effects of charge relaxation cannot be ignored, as the width of the jet is determined partly by the charge relaxation rate (Mestel 1994*a*; other papers in this special journal issue are also relevant). In a previous work, the author investigated the effects of charge, tangential field and a finite charge relaxation rate for a high Reynolds number flow (Mestel 1994*b*). There, for mathematical convenience, axisymmetric perturbations were considered mainly, so that the perturbed flow was irrotational apart from a thin surface layer. In this paper, the opposite inertialess limit will be considered, for all wavenumbers. The calculation is here much more involved, and without the computer algebra package *Mathematica* the general case would have been difficult to contemplate.

The observations suggest that the jet may do one of three things: it may remain stable for a surprising distance, it may break up to long axisymmetric disturbances as it does when subject purely to capillary forces, or it may perform a sinuous whipping motion, an instability quintessentially electrohydrodynamic (Taylor 1969; Magarvey & Outhouse 1962; Huebner 1969). Our aim in this paper is to determine under which circumstances each form of behaviour is to be expected when the Reynolds number is low. We formulate the stability problem for linear, temporal modes in §2 subject to the electrical stresses discussed in §3. The axisymmetric and sinuous modes are discussed in detail in §4 and §5 respectively. It is found that a suitable combination of q and E can stabilize the axisymmetric modes against both capillary and charge relaxation instabilities. The sinuous modes, however, are strange in that the extremely long waves are violently unstable. This feature was found by Saville (1971*b*) when merely surface charge is present for a perfect conductor. It can be considered as an artefact of the geometry, in that the two-dimensional Stokes equations do not behave well. They respond with a massive amplification of the weak repulsive force caused by the infinitesimal sinuous displacement of the surface charge. Inclusion of a small inertial term, however, controls the singularity. Alternatively, as the instability requires the logarithm of the vertical wavelength to be large, we can arbitrarily ignore sufficiently long disturbances, relying on some neglected physics to control these maverick modes. Once this is done, parameter ranges can be found which stabilize the sinuous waves. Overlapping stability regions for both sinuous and axisymmetric modes then exist in some parameter ranges.

Another curious result is that there is a critical permittivity ratio $\lambda = 3$, below which sinuous relaxation stabilities can exist at low Reynolds number, a feature which has not previously been noticed. Drozin (1955) reported difficulty in producing jets at low permittivity, although Jones & Thong (1971) found that these do exist for small enough relaxation times. However, a high permittivity exerts a stabilizing influence for all modes and Reynolds numbers and this theoretical result may not be related to those experiments.

The manner in which the inclusion of small inertial terms can limit both the wavelength and the growth rate of the most unstable sinuous mode is examined in §6. In §7, modes with higher azimuthal wavenumber are briefly considered and we conclude in §8.

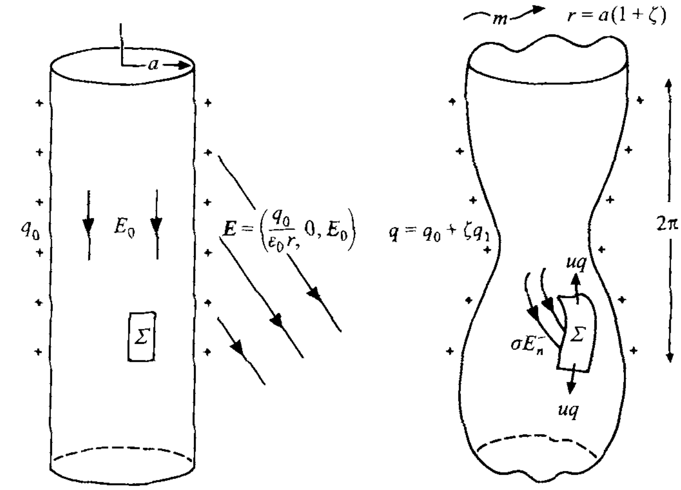

FIGURE 1. The perturbation of the jet.

2. Formulation of the problem

We consider a liquid cylinder of radius a, density ρ_0, viscosity μ_0, electrical permittivity ε and conductivity σ, surrounded by an insulating gas of vacuum permittivity ε_0. A uniform axial electric field $(0, 0, E_0)$ with respect to cylindrical coordinates (r, θ, z) exists inside and outside the cylinder, as in figure 1. The interface carries a uniform electric charge density q_0, with a surface tension γ_0. We non-dimensionalize length with respect to a, time with respect to the capillary scale $a\mu_0/\gamma_0$ and mass with respect to $a^2\mu_0^2/\gamma_0$. The relevant parameters are

$$\rho = \frac{a\rho_0\gamma_0}{\mu_0^2}, \qquad q^2 = \frac{q_0^2 a}{\varepsilon_0\gamma_0}, \qquad E^2 = \frac{\varepsilon_0 E_0^2 a}{\gamma_0}, \qquad \gamma = 1. \tag{2.1}$$

The low Reynolds number assumption is that $\rho \ll 1$. The parameter γ is retained to identify where appropriate the action of surface tension. Also important are two electrical parameters, the permittivity ratio λ and charge relaxation time τ:

$$\lambda = \frac{\varepsilon}{\varepsilon_0} \quad \text{and} \quad \tau = \frac{\varepsilon}{\sigma}\frac{\gamma_0}{a\mu_0}. \tag{2.2}$$

This non-dimensionalization is more appropriate for the low Reynolds number flows to be considered here than that used by Mestel (1994*b*) for the high Reynolds number problem. Here we shall usually assume that $\rho = 0$.

Before we forget about inertia, however, we should recognize that in reality the charge on the jet surface will be pulled by the tangential field and thus that the jet may accelerate. If the inlet velocity is fixed, then the resultant thinning of the jet could invalidate the model. If we permit the entire jet to accelerate, then its vertical velocity is

$$v_z(r, t) = v_0 + \tfrac{1}{2}qE(r^2 - 1) + \left(\frac{2qE}{\rho} + g\right)t, \tag{2.3}$$

where g denotes a suitably non-dimensional gravity. We could restrict ourselves to a scenario in which the imposed tangential stress exactly balanced gravity, so that the net acceleration is zero. More usefully, we can regard this as a physical limitation on the vertical lengthscales permitted by the theory, as discussed in Mestel (1994*b*).

We shall work in a (possibly accelerating) frame with the jet surface at rest and consider perturbations so that the surface S is defined by

$$r = (1 + \zeta) \qquad \text{where} \quad \zeta = \delta e^{ikz + im\theta + st}, \tag{2.4}$$

with $\delta \ll 1$ and s being the (complex) growth rate of the disturbance. The wavenumbers k and m are real and positive but otherwise arbitrary save that m must be integral. The unit normal to the surface, $\hat{\boldsymbol{n}}$, is then

$$\hat{\boldsymbol{n}} = (1, -im\zeta, -ik\zeta). \tag{2.5}$$

The perturbed pressure p and velocity $\boldsymbol{u}$ take the form

$$p = p_0 + \gamma + \hat{p}\,, \qquad \boldsymbol{u} = (0, 0, \tfrac{1}{2}qE(r^2 - 1)) + \hat{\boldsymbol{u}}\,,$$

where p_0 is the non-dimensionalised constant atmospheric pressure, while the perturbations $\hat{p} = \zeta p(r)$ and $\hat{\boldsymbol{u}} = (u_r(r), u_\theta(r), u_z(r))\zeta$ obey the Stokes equations

$$\nabla \cdot \hat{\boldsymbol{u}} = 0 \qquad \text{and} \quad \nabla \hat{p} = \nabla^2 \hat{\boldsymbol{u}}. \tag{2.6}$$

In component form these are

$$(ru_r)' + imu_\theta + ikru_z = 0 \tag{2.7}$$

and

$$\left.\begin{aligned} r^2u_r'' + ru_r' - (m^2+1)u_r - k^2r^2u_r + 2imu_\theta &= r^2p', \\ r^2u_\theta'' + ru_\theta' - (m^2+1)u_\theta - k^2r^2u_\theta - 2imu_r &= imrp, \\ r^2u_z'' + ru_z' - u_z - k^2r^2u_z &= ikr^2p. \end{aligned}\right\} \tag{2.8}$$

The general solution regular at $r = 0$ can be written

$$\left.\begin{aligned} p &= PI_m, \\ u_r &= P\left[krI_m - (m+2)I_{m+1}\right] + BI_{m-1} + CI_{m+1}, \\ u_\theta &= iP(m+2)I_{m+1}/(2k) + iBI_{m-1} - iCI_{m+1}, \\ u_z &= \tfrac{1}{2}iPrI_m + i(B+C)I_m, \end{aligned}\right\} \tag{2.9}$$

where I_m is a modified Bessel function of argument kr. The constants P, B and C are determined by matching the surface stresses, which we write in the form

$$2u_r' - p = T_n, \qquad u_z' + iku_r = T_z, \qquad u_\theta' + \frac{im}{r}u_r = T_\theta. \tag{2.10}$$

The growth rate then follows from the kinematic surface condition

$$s = u_r. \tag{2.11}$$

When the imposed surface stresses (T_n, T_θ, T_z) are independent of $\boldsymbol{u}$ and s, the above procedure gives a unique growth rate in a straightforward manner. For example, in the Plateau problem (1873), where surface tension alone acts, we have

$$T_\theta = T_z = 0\,, \qquad T_n = \gamma(1 - m^2 - k^2). \tag{2.12}$$

The growth rate for $m = 0$ turns out to be

$$s = \tfrac{1}{2}\gamma\, \frac{(1-k^2)r_0^2}{k^2 - r_0^2 - k^2r_0^2} \qquad \text{where} \quad r_0 = \frac{I_1(k)}{I_0(k)}. \tag{2.13}$$

All wavenumbers in the range $0 < k < 1$ are unstable for this inertialess calculation, with the highest growth rate occurring as $k \to 0$, when $r_0 \sim \frac{1}{2}k$ and $s \to \frac{1}{6}\gamma$. These results were given by Rayleigh (1892), who comments that Plateau (1873) was more concerned with viscous than inertial effects, as are we here. The inclusion in the calculation of a weakly viscous surrounding fluid limits the size of the most unstable wavelength (Tomotika 1935), as indeed does a small amount of inertia (see §6).

The sinuous ($m = 1$) mode is also of interest, for which

$$s = \gamma \frac{-2k + 6r_1 - 2k^2 r_1 + 7kr_1^2 - k^3 r_1^2 + 2k^2 r_1^3}{2k(k^2 - 4kr_1 + 3r_1^2 - k^2 r_1^2 + 2kr_1^3)} \quad \text{with} \quad r_1 = \frac{I_2(k)}{I_1(k)}. \tag{2.14}$$

All k values are stable in this case, the long waves especially so, since as $k \to 0$ we have $r_1 \sim \frac{1}{4}k$ and

$$s \sim -\frac{4}{3k^2}\gamma = \frac{4T_n}{3k^4}. \tag{2.15}$$

We note the violence of the response to such modes and in particular the sensitivity to the sign of T_n. The importance of this effect will become clear in §6. Similar expressions can be found for modes with higher values of m, which are always stable. For large values of k all modes are stable with $s \simeq -k\gamma/2$ for all $m \ll k$.

When electrical forces act the surface stresses are more complicated, especially since for finite electrical conductivity they depend explicitly on both s and u_r. We discuss these in the next section.

3. The surface stresses

The electric field perturbation was calculated in detail in Mestel (1994*b*) and here we summarize the results. Using the suffices + and − to denote outside and inside the jet respectively, the perturbed electric potentials $\phi^{\pm}$ take the form

$$\left.\begin{aligned} \phi^+ &= -Ez - q\ln r + C\zeta\, K_m(kr), \\ \phi^- &= -Ez \qquad\qquad + D\zeta\, I_m(kr). \end{aligned}\right\} \tag{3.1}$$

The coefficients C and D of the modified Bessel functions I_m and K_m can be found from the boundary conditions on $r = 1 + \zeta$. These are the continuity of potential,

$$\phi^+ = \phi^-, \tag{3.2}$$

and the conservation of charge

$$-\frac{1}{\tau}\frac{\partial \phi^-}{\partial n} = \zeta(sq_1 - qR), \tag{3.3}$$

where q_1 is the surface charge perturbation and R is the rate of dilation of surface elements

$$\hat{\boldsymbol{n}} \cdot [(\hat{\boldsymbol{n}} \cdot \nabla)\boldsymbol{u}] \equiv \zeta R = \zeta\left[-\mathrm{i}kqE + u_r'(1)\right]. \tag{3.4}$$

It is the relation (3.3) which complicates the problem greatly. In balancing the flow of charge on a surface element Σ, allowance must be made for electrical conduction, the growth of the perturbation and the advection of charge due to motion of the surface, as indicated in figure 1. Unless $\tau = 0$, the surface stresses thus depend explicitly both on the growth rate s and on the velocity perturbation u_r. We write the stresses in the form

$$(T_n, T_\theta, T_z) = (T_n^0, T_\theta^0, T_z^0) + R(T_n^R, T_\theta^R, T_z^R), \tag{3.5}$$

where from Mestel (1994*b*)

$$(T_n^R, T_\theta^R, T_z^R) = \frac{\tau}{W}\Big(-\mathrm{i}qE(\lambda - 1) - \tilde{K}q^2,\ -\mathrm{i}q^2\frac{m}{k},\ qE(\lambda\tilde{I} - \tilde{K}) - \mathrm{i}q^2\Big), \tag{3.6}$$

with

$$W = \lambda\tilde{I} + s\tau(\lambda\tilde{I} - \tilde{K}) , \qquad \tilde{K} = K_m'(k)/K_m(k) , \qquad \tilde{I} = I_m'(k)/I_m(k) \tag{3.7}$$

and

$$\left.\begin{aligned}
T_n^0 &= \gamma(1-k^2-m^2) + \frac{1}{W}\Big[kE^2(\lambda-1)[s\tau(1-\lambda)-\lambda] - \lambda q^2\tilde{I}(1+k\tilde{K})(1+s\tau) \\
&\quad + E^2q^2k(1-\lambda)\tau + \mathrm{i}qE[-3kW + k\tilde{K}\lambda - (\lambda-1)s\tau + k\tilde{K}q^2\tau]\Big] , \\
T_\theta^0 &= \frac{im}{kW}\left[-(1+k\tilde{K})q^2s\tau + \mathrm{i}kqE[\lambda + q^2\tau + (\lambda-1)s\tau]\right] , \\
T_z^0 &= \frac{\mathrm{i}}{W}\Big[kE^2\lambda(\tilde{K}-\tilde{I}) - q^2s\tau(1+k\tilde{K}) + E^2q^2\tau k(\tilde{K}-\lambda\tilde{I}) \\
&\quad - \mathrm{i}qE[-\lambda(\tilde{I} + k + k\tilde{I}\tilde{K}) - kq^2\tau + ks\tau(1-\lambda) - W]\Big].
\end{aligned}\right\} \tag{3.8}$$

The relations (3.4–3.8) can be substituted into (2.9) and (2.10) and the resulting system of equations solved to find the growth rate s, taking note that R depends on P, B and C. With $\tau \neq 0$, we obtain a quadratic equation with complex coefficients for s, which we write for given m as

$$a_m s^2 + b_m s + c_m = 0, \tag{3.9}$$

with the coefficients having no common factors or denominators. We find that we can define real and imaginary parts such that $a_m = a_r \geqslant 0$, $b_m = b_r + \mathrm{i}b_i$ and $c_m = c_r + \mathrm{i}c_i$. Then the conditions for stability are

$$b_r \geqslant 0 \qquad \text{and} \qquad c_r b_r^2 + b_i b_r c_i - a_r c_i^2 \geqslant 0. \tag{3.10}$$

The algebra required is exceedingly nasty, even with the help of *Mathematica*. The expressions for b_m and c_m cover several pages, and the program required some guiding to prevent storage overflow during the calculation. Fortunately the problem is linear and may be subdivided into algebraically tractable portions. Agreement was found with all previous works which can be obtained as limits of the general case.

The coefficient a_m is more manageable, being $\tau(\lambda\tilde{I} - \tilde{K})$ multiplied by a real polynomial in k, m and $r_m = I_{m+1}(k)/I_m(k)$. It never vanishes except when $\tau = 0$ so that (3.9) has two roots, which is what one would expect on physical grounds. As $\tau \to 0$, the important root is $s \simeq -c_m/b_m$, the other one being

$$s \simeq -\frac{b_m}{a_m} \simeq -\frac{1}{\tau}\frac{\lambda\tilde{I}}{\lambda\tilde{I} - \tilde{K}} < 0. \tag{3.11}$$

Thus for small values of τ, charge relaxation is not necessarily destabilizing at low Reynolds number. This contrasts with the high Reynolds number behaviour, when all modes are unstable for $\tau > 0$ (Saville 1971*a*; Mestel 1994*b*).

For any values of the physical parameters q, E, τ, λ, we can now evaluate the possible growth rates of the disturbance for any k and m. Further, we can calculate the wavenumber k corresponding to the growth rate with the greatest real part. When appropriate, we can find conditions on the parameters such that this value is zero, delimiting stable and unstable regions of parameter space. We begin by considering the axisymmetric disturbances, which are dominant in the absence of electric effects.

4. Axisymmetric modes

When surface tension alone acts, we saw in §2 that axisymmetric modes are unstable for $0 < k < 1$, with the most unstable being the long waves, $m = 0$ and $k \to 0$. This contrasts with the higher Reynolds number behaviour when the most unstable mode has $k \approx 0.7$ (Rayleigh 1879). The behaviour of the axisymmetric long waves for the electric jet is thus of primary interest.

The growth rate s is given by $a_0 s^2 + b_0 s + c_0 = 0$. As $k \to 0$ with none of τ, E and q zero, we find

$$a_0 \sim 3\tau, \qquad b_0 \sim -\tfrac{1}{2}\tau(\gamma + 4q^2 \log k) \qquad \text{and} \quad c_0 \sim E^2(\lambda + q^2\tau) + O(k^2 \log k). \tag{4.1}$$

An $O(1)$ term accompanies the logarithm as usual, and throughout this paper $(\log k)$ is used to denote $(\ln k - 0.11593)$. This constant term might be modified if a different electrical boundary condition at infinity were deemed appropriate for a specific problem. Since $a_0 > 0$ and $c_0 > 0$, the condition for stability is $b_0 \geqslant 0$. The logarithm ensures that this is satisfied for small enough k for any $q > 0$, so that any amount of surface charge can stabilize the very long waves. If q is small, however, as k increases from zero the logarithm is soon essentially $O(1)$ and (4.1) may still be a valid approximation for some small wavenumbers such that $b_0 < 0$, with resultant instability. Furthermore, if the theory is to apply in a practical situation, it is likely that large values of $(\log k)$ with the attendant very large lengthscales will be of limited importance. In fact we will see that $O(1)$ values of q are necessary to stabilize all axisymmetric modes.

When $q = 0$, we have the problem considered by Saville (1970, 1971*a*). The long waves are unstable to charge relaxation overstabilities with

$$s = \tfrac{1}{12}\gamma \pm \left(\gamma^2/144 - E^2\lambda/3\tau\right)^{1/2}. \tag{4.2}$$

As τ decreases or E increases the capillary instability becomes oscillatory with half the growth rate of that of the non-electric jet. This is illustrated in figure 3 of Saville (1971*a*) (there is a typographical error in that figure, 0.25 should read 2.5).

When $\tau = 0$, the oscillatory instabilities are suppressed. We have $b_0 \simeq -\tfrac{3}{2}\lambda k^2 \log k$ and

$$s \sim \tfrac{2}{3}E^2/(k^2 \log k) + \tfrac{1}{6}(\gamma - q^2) \tag{4.3}$$

to leading order. Any tangential field suffices for stability of long waves, as does a surface charge $q^2 > \gamma$.

When $E = 0$, to leading order as $k \to 0$ we have $c_0 \sim \tfrac{1}{4}k^2 \log k(\gamma - q^2)(\lambda + q^2\tau)$. A weak instability occurs if $\gamma > q^2$ and an $O(1)$ one if $\gamma > -4q^2 \log k$.

Naively, it might be expected that if very large values of k are stable as well as very small values, then there is a reasonable chance of overall stability. We therefore consider also the limit $k \to \infty$. Surface tension has a strong damping effect on very short wavelengths, but if q and E are much greater than unity, an unstable band of modes might occur for large but not enormous values of k. If we set $\tau = 0$ for simplicity and let $k \to \infty$ we find

$$s \sim -\tfrac{1}{2}\gamma k + \tfrac{1}{2}\left[q^2 - 4iqE - (\lambda - 1)E^2\right]. \tag{4.4}$$

Stability for large values of k is thus guaranteed provided

$$(\lambda - 1)E^2 \geqslant q^2. \tag{4.5}$$

For $\tau > 0$ a more complicated relation holds, with instability occurring at a smaller

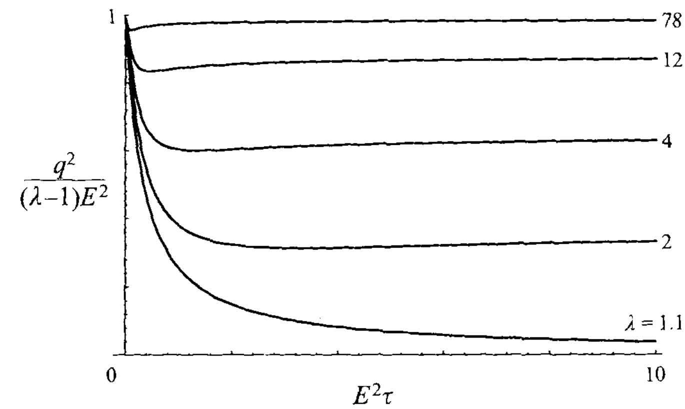

FIGURE 2. Short waves, $k \gg 1$ with $q, E \gg \gamma$. Critical value of $q^2/[(\lambda-1)E^2]$ as a function of τE^2 for $\lambda = 1.1, 2, 4, 12, 78$. Regions above these curves are unstable.

value of τ. In figure 2 the critical charge destabilizing the short waves when surface tension is negligible is plotted against τ for $\lambda = 1.1, 2, 4, 12, 78$. For physical reasons $\lambda \geqslant 1$; $\lambda \simeq 2$ for many organic solvents while $\lambda = 78$ corresponds to water. It should however be borne in mind that except for very small jet diameters, the low Reynolds number assumption $\rho \ll 1$ will not hold for aqueous jets. With $\lambda = 1$ any $q > 0$ is unstable, while for $\lambda \gg 1$ (4.5) is a reasonable approximation for any τ. There is a minimum of q at a finite value of τ. This tendency for small values of λ to require small τ for stability agrees with the observations of Jones & Thong (1971).

We now consider the full problem for the axisymmetric modes. First, we fix $\tau = 0.1$ and for different values of λ we calculate the neutral stability curves in the (q, E)-plane, by finding when the maximum growth rate over all k is zero. The results for the four values $\lambda = 2, 4, 12, 78$ are shown in figure 3. Each neutral stability curve has a lower branch, below which the long waves (small k) are unstable. As described above, these are not the asymptotically small values of k, but rather some intermediate small value. Each curve also has a left-hand branch, above which some moderately large wavenumber is unstable, although not so large that surface tension dominates. These two branches meet at a sharp corner, at which two different values of k have the same real part of s and the most unstable wavenumber is discontinuous. One might expect the left-hand branches of these curves to asymptote to a gradient given by figure 2, but this need not occur if Re(s) is maximum at some finite k-value.

As the permittivity of the jet λ is increased, a smaller electric field has a stronger polarizing effect, and so smaller values of E are necessary for stability. Regions to the top-right of figure 3 are stable to all $m = 0$ modes.

We now consider the effect of varying the conductivity of the jet, by fixing the permittivity $\lambda = 2$ and allowing τ to take various values. The resulting stability regions are shown in figure 4. When $\tau = 0$, tangential field alone can stabilize all axisymmetric modes, but for $\tau > 0$ surface charge is required to prevent the over-relaxation instability of Melcher & Schwarz (1968) and Saville (1971a). The neutral stability curves have two branches once more, the lower one indicating stabilization of the long waves, and the upper one shorter waves. As τ increases, the lower branch surprisingly turns over, but always for given q there is a single critical value of E above which all axisymmetric modes are stable, so that an increase in tangential field is never destabilizing.

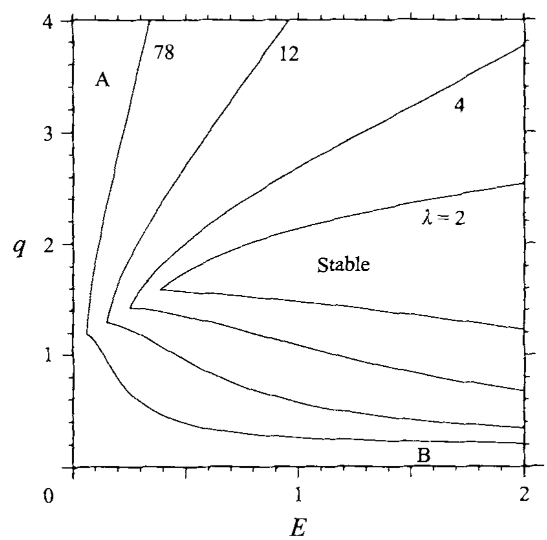

FIGURE 3. Axisymmetric modes for $\tau = 0.1$, $\lambda = 2, 4, 12, 78$. Regions to the right are stable for all k. Near A and B, large and small k respectively are unstable.

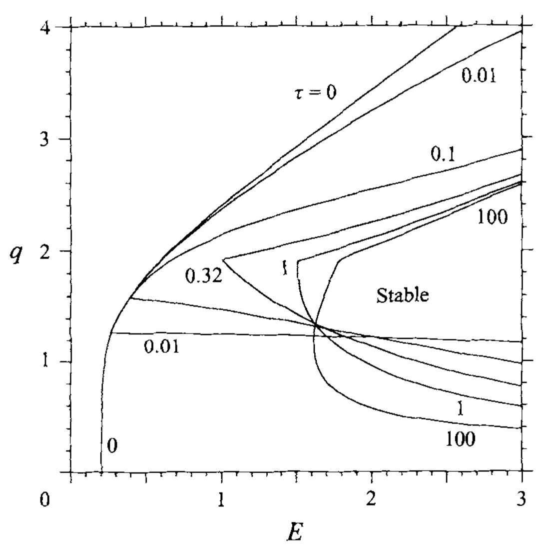

FIGURE 4. Axisymmetric modes for $\lambda = 2$, $\log_{10} \tau = -\infty, -2, -1, -0.5, 0, 2$.

5. Sinuous modes ($m = 1$)

We have seen that at low Reynolds number, suitable values of both surface charge and tangential field can stabilize all axisymmetric modes. Nevertheless, experiments have shown that the sinuous modes can also be unstable for charged jets and Saville (1971*b*) has demonstrated this theoretically when $E = 0$ and $\tau = 0$. He found that once more the long waves $k \to 0$ were the most unstable, and we begin by investigating this limit. The short waves, with $k \to \infty$ are described once more by figure 2.

Setting $m = 1$ and letting $k \to 0$, the coefficients of the stability quadratic are

$$\left.\begin{aligned} a_1 &\sim \tfrac{3}{4}k^3(1+\lambda)\tau, \\ b_1 &\sim 4\mathrm{i}qE\tau(1+\lambda) + k\tau\left[E^2(\lambda-1)^2 + \gamma(1+\lambda) + (1+\lambda)q^2\log k\right] + \tfrac{3}{4}k^3\lambda, \\ c_1 &\sim 2\mathrm{i}qE(2\lambda + q^2\tau) + \tfrac{1}{2}k\left[(2\lambda+q^2\tau)(q^2\log k + \gamma) + 2\lambda E^2(\lambda-3)\right]. \end{aligned}\right\} \quad (5.1)$$

There is also a term of order $(kq^2E^2\tau)$ in c_1, but for no parameter values is this important for small k. Once more, $(\log k)$ denotes $(\ln k - 0.11593)$ where significant.

One of the roots given by (5.1) is stable if $q\tau \neq 0$,

$$s \simeq -\frac{c_1}{b_1} = -\frac{2\lambda + q^2\tau}{2\tau(1+\lambda)} + O(k), \quad (5.2)$$

but the other is

$$s \simeq -\frac{b_1}{a_1} = -\frac{16\mathrm{i}qE}{3k^3} - \frac{4}{3k^2}\left(q^2\log k + \gamma + E^2\frac{(\lambda-1)^2}{(\lambda+1)}\right). \quad (5.3)$$

The leading term is rapidly oscillating, but at next order we have the stability condition

$$q^2\log k + \gamma + E^2\frac{(\lambda-1)^2}{(\lambda+1)} \geqslant 0. \quad (5.4)$$

As we saw in §2, this is satisfied if $q = E = 0$. However, with $q \neq 0$, the logarithm which guaranteed stability for the axisymmetric mode here ensures that sufficiently long waves are unstable. Furthermore, the instability has a massive growth rate $\mathrm{Re}[s] \sim 1/k^2$.

This strange effect was pointed out by Saville (1971*b*) for the case with $E = \tau = 0$. It survives the presence of a tangential electric field and the resultant motion of the base state of the jet. The main difference is the presence of the neutrally stable higher-order term, which may mitigate the effect when more physics is included in the problem, as in §6.

At first glance, the instability of this mode is paradoxical. After all, $m = 1$ and $k = 0$ corresponds to a lateral displacement of the entire jet, which from physical grounds is obviously neutrally stable. Yet the presence when $k = 0$ of the eigen-solution of (2.5) with $s = 0$, $(u_r, u_\theta, u_z) = A(1, \mathrm{i}, 0)$ can be regarded as being responsible for the exceptional behaviour of small k for $m = 1$. Because of the cancellation at leading order, it is the higher-order terms which must deal with the stress perturbations, which therefore requires a high value of A in the leading-order flow.

Formally, we have found that for any $q > 0$ the jet is highly unstable, and the most unstable modes are the sinuous long waves. However these may require the logarithm of the vertical lengthscale to be large. We know that there are physical limitations on the vertical lengthscales admissable by the theory, whether because of the neglect of inertia, the finite length of the experimental apparatus, the loss of validity due to acceleration and thinning of the jet or non-uniformities in the imposed electric field. The existence or otherwise of smaller wavelength instabilities is likely to be of greater relevance in practice than the logarithmically asymptotic instability of (5.3). We shall therefore impose an arbitrary limit on the size of k, requiring

$$k \geqslant k_{min} = \mathrm{e}^{-4} \simeq 0.0183. \quad (5.5)$$

We can then investigate the stability of modes with $k \geqslant k_{min}$. First we consider what happens if $q\tau = 0$.

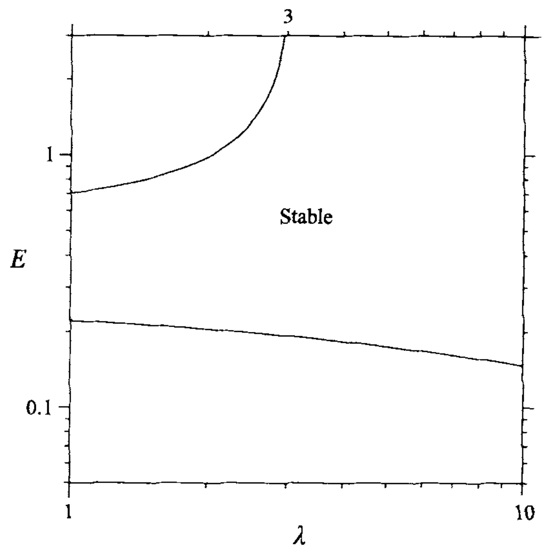

FIGURE 5. Uncharged jet for $\tau \to 0$ and various E, λ. Below the lower curve $m = 0$ modes are unstable; above the higher curve, $m = 1$ modes are unstable for $\lambda < 3$.

If $\tau = 0$ and subsequently $k \to 0$, there is only one root,

$$s = -\frac{16iqE}{3k^3} - \frac{4}{3k^2}\left[q^2 \log k + \gamma + E^2(\lambda - 3)\right]. \tag{5.6}$$

This is similar to (5.3), although the tangential field term is different. The two limits $\tau \to 0$ and $k \to 0$ are not interchangeable.

When $q = 0$, the destabilizing logarithm vanishes, and the large root given by (5.3) is stable. However, the other root is no longer given by (5.2), but rather

$$s = -\left(\frac{\lambda}{\tau}\right) \frac{\gamma + E^2(\lambda - 3)}{E^2(\lambda - 1)^2 + \gamma(\lambda + 1)}. \tag{5.7}$$

Thus we find that the long waves are stable unless the permittivity ratio $\lambda < 3$ and

$$\gamma < E^2(3 - \lambda). \tag{5.8}$$

There is no obvious physical reason why the charge relaxation instability should manifest itself only at low permittivities for the sinuous modes. Saville (1971*a*) considers the $q = 0$ case but incorrectly states that "with viscous effects dominant ... only axisymmetric motions can be unstable," presumably influenced by his extensive study of the case $\lambda = 78$, appropriate to water. In figure 5 we plot with logarithmic axes the stability region of an uncharged jet for $\tau \to 0$. The lower curve represents the value of E for given λ necessary to stabilize all axisymmetric modes. The upper curve gives the maximum value of E below which all sinuous modes are stable. It exists only for $\lambda < 3$, as it happens that the long waves appropriate to (5.8) are the most unstable.

We now return to the general case with the restriction (5.5) applied to k. In figure 6 we plot the neutral stability curves for $\tau = 0.1$ and the same values of λ as in figure 3, $\lambda = 2, 4, 12, 78$. The value $q = 0.5$ corresponds to where $q^2 \log(k_{min}) + \gamma = 0$, so that all the curves start close to this value when $E = 0$. At higher values of λ, the tangential field in (5.3) becomes significant more quickly as shown in the graphs.

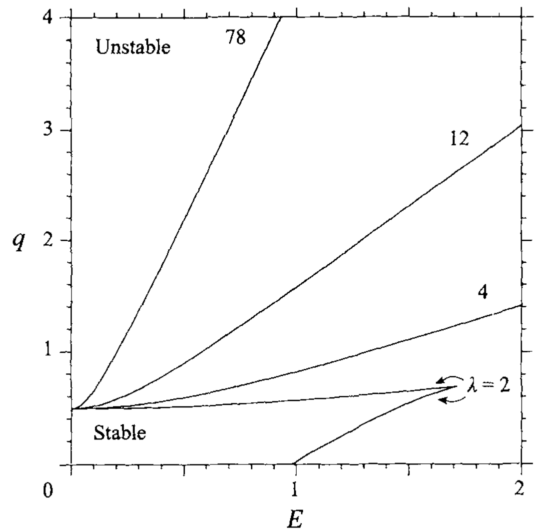

FIGURE 6. Sinuous modes for $\tau = 0.1$, $\lambda = 2, 4, 12, 78$, with $k \geqslant e^{-4}$ restriction. Regions including the origin are stable. For $\lambda > 3$ increasing E is stabilizing but for $\lambda = 2$ there is a maximum stable E-value.

Only one of the illustrated λ-values is less than 3, and for this value there is another branch to be followed. When $E \simeq 1$ and $q = 0$ for $\lambda = 2$ the over-relaxation instability appears, requiring surface charge to control it. These two branches for $\lambda = 2$ intersect, to the right of which unstable k-values exist.

In figure 7, we fix $\lambda = 2$ and vary the charge relaxation time τ. For $\tau = 0$ the stable region is approximately $E^2 + 4q^2 \leqslant 1$, for the chosen value of k_{min}. For $\tau > 0$, there are two neutral branches for each τ value. The position of the upper branch is almost independent of τ, as it is determined essentially by k_{min}. The lower branch weakens as τ increases, giving a wider range of stable (q, E) values.

Figures 3, 4, 6 and 7 may be combined to find parameter values for which the jet is completely stable, provided $k \geqslant k_{min}$. Polar liquids (high λ) have larger stability regions at low Reynolds number. For values of $\lambda < 3$ and intermediate values of τ (such as $\tau = 0.1$, $\lambda = 2$) the jet is always unstable to one mode or another.

In the next section we examine the manner in which a small amount of inertia can limit the size of the wavelength and growth rate of the most unstable sinuous mode.

6. The limiting effect of inertia

As $k \rightarrow 0$ with $m = 1$ we have seen that the Stokes equations behave strangely, giving rise to large velocities and growth rates in response to a small normal stress. This behaviour is connected to the neutral stability of the infinitesimal lateral displacement corresponding to $k = 0$, $m = 1$. In a real problem, the singularity as $k \rightarrow 0$ is limited either by physical constraints invalidating the model for large enough vertical wavelengths, or by the presence of an external viscous fluid or by neglected inertial terms becoming important for any non-zero Reynolds number. To illustrate how this occurs we begin by considering Saville's (1971*b*) problem of a perfect conductor carrying surface charge only, so that $E = 0$ and $\tau = 0$. The unperturbed jet velocity is then zero, and the only non-zero surface stress perturbation is T_n.

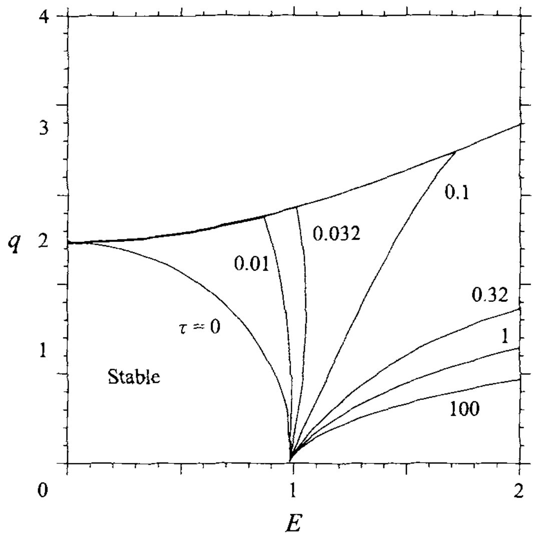

FIGURE 7. Sinuous modes for $\lambda = 2$, $\log_{10}\tau = -\infty, -2, -1.5, -1, -0.5, 0, 2.$ with $k \geqslant e^{-4}$. Regions including the origin are stable.

Including the inertial term $\rho s \boldsymbol{u}$ in (1.6) and taking the limit $k \to 0$, we obtain

$$\rho s^2 + \tfrac{3}{4}k^4 s = -q^2k^2 \log k - \gamma k^2 \qquad \text{with} \quad \rho \ll 1. \tag{6.1}$$

This relation agrees with an appropriate limit of equation (9) of Saville (1971*b*). It assumes that $(k^2 + \rho s)$ may be treated as small. Modes with sufficiently small values of k such that $-\log k > \gamma/q^2$ are unstable. As k decreases to zero, the growth rate s given by (6.1) increases to a maximum until the inertial term becomes important. It then decreases to zero, remaining positive all the while. Neglecting surface tension to leading order, the most unstable wavenumber as $\rho \to 0$ is given by

$$\frac{\partial s}{\partial k} = 0 \qquad \text{when} \quad k^6 = -\tfrac{8}{9}\log k\; q^2\rho, \tag{6.2}$$

for which the growth rate is

$$s = \left(\frac{L^2q^4}{3\rho}\right)^{1/3}, \tag{6.3}$$

where L is the value of $(-\log k)$ given by (6.2).

We now consider the effect of the other stress components. When $\rho = 0$ the growth rate for $k \to 0$, $m = 1$ is

$$s = \frac{4}{3k^4}(T_n - \mathrm{i}T_\theta + \mathrm{i}kT_z) \tag{6.4}$$

illustrating the violent reaction to a non-zero sinuous stress component. When neither E nor q is zero, we saw in (5.3) that T_z gives rise to a rapid oscillation as $k \to 0$ with $m = 1$. When inertia is included, the equation corresponding to (6.1) when $\tau = 0$ is

$$\rho s^2 + \tfrac{3}{4}k^4 s = -4\mathrm{i}kqE - k^2[q^2\log k + \gamma + E^2(\lambda - 3)]. \tag{6.5}$$

Now the imaginary term originating from T_z is the first to interact with the inertial term as $k \to 0$. Asymptotically for small ρ we can neglect the other three terms on the right-hand side of (6.5) to find that the maximum of the growth rate, $\mathrm{Re}(s)$, occurs when

$$\frac{\partial}{\partial k}\mathrm{Re}[s] = 0 \qquad \text{for} \quad k^7 = 0.253\,\rho q E \tag{6.6}$$

and takes the value

$$\mathrm{Re}(s) = 1.1167\left(\frac{(qE)^4}{\rho^3}\right)^{1/7}. \tag{6.7}$$

Both the maximum growth rate and corresponding wavenumber are larger as $\rho \to 0$ than those given by (6.2) and (6.3), although for moderate values of ρ the difference is not so great. With $\rho = 0.01$ and $q = 1$, for example, the value of k given by (6.2) is only $k \simeq 0.45$, for which $L < 1$ even! In (6.3) we then have $s \simeq 3.04$. If in addition we take $E = 1$, then (6.6) and (6.7) give instead $k \simeq 0.43$ and $\mathrm{Re}[s] \simeq 8$. In practice with even a small amount of inertia the most unstable wavelength will not be so very large.

A similar calculation can be performed for axisymmetric modes, so that if $E = 0 = \tau$ we obtain

$$\rho s^2 + 3k^2 s + \tfrac{1}{2}k^2(q^2 - \gamma) = O(k^4). \tag{6.8}$$

If $\gamma > q^2$, the most unstable mode has $k^2 = O(\rho)$. For $k^2 \ll \rho$, or for the sinuous case k much smaller than given by (6.2) or (6.6), the analysis of this section would require modification, as the limits $k \to 0$ and $\rho \to 0$ are not interchangeable.

It is clear that while finite inertia controls the singularity, a sinuous instability can be expected if there exists a sufficient vertical length of jet for it to be realized. Presumably the inclusion of an external viscous fluid along the lines of Tomatika (1935) would also lead to finite values of the growth rate and maximally unstable wavelength.

7. Other limits

We conclude with a brief survey of other limits of the full expressions. First of all, we consider the long waves for $m \geqslant 2$, when we find

$$\left.\begin{aligned} a_m &\sim 4(\lambda+1)(m+1)\tau, \\ b_m &\sim 4[(m+1)\lambda + mq^2\tau] + 2m\tau(\lambda+1)(\gamma(m+1)-q^2), \\ c_m &\sim m(\gamma(m+1)-q^2)(2\lambda+q^2\tau). \end{aligned}\right\} \tag{7.1}$$

The tangential field is not important as $k \to 0$ for $m > 1$. These modes are stable if the surface charge is not too large,

$$q^2 \leqslant (m+1)\gamma. \tag{7.2}$$

This is the same condition as found by Saville (1971*b*) even though here $\tau \neq 0$ and $E \neq 0$. The constraint (7.2) is less severe than that posed by the sinuous modes unless a large value of k_{min} is taken. No cases were found when the most unstable mode had $m > 1$.

Taking the limit $\lambda \to \infty$ is not possible physically, but it illustrates the stabilizing tendencies of the tangential field for highly polar liquids. Both roots of the quadratic (3.9) are stable for all m in this limit, one satisfying $\mathrm{Re}[s] \to -1/\tau$ and the other proportional to λ.

The very poor conductor limit $\tau \to \infty$ does not result in much simplification of the problem. Surface charge is still redistributed in response to the perturbation but by advection rather than conduction. For the long sinuous waves this process is very fast. In the absence of surface charge, the sinuous instability for $\lambda < 3$ given by (5.7) remains as $\tau \to \infty$, but becomes very slow.

8. Concluding remarks

In this paper we have shown that interactions among surface charge, tangential field and charge relaxation can stabilize a low Reynolds number jet, provided some limitation on the axial lengthscale can be imposed. Loosely speaking, the surface charge stabilizes axisymmetric long waves, but destabilizes $O(1)$ waves. The simultaneous action of a large enough electric field stabilizes all axisymmetric modes. Some surface charge is necessary to suppress the charge relaxation instabilities associated with non-zero E and τ.

The presence of surface charge does, however, excite sinuous instabilities. These are particularly acute in the limit of zero Reynolds number, when the growth rates of long waves are unbounded. A small amount of inertia limits the singularity, but nevertheless a band of sinuous instabilities can be expected for a moderate surface charge. If for some unspecified reason the logarithmically long waves can be neglected, however, a small amount of surface charge with a larger tangential field can stabilize the jet against all disturbances, especially for high permittivities. For permittivities $\lambda < 3$, a sinuous charge relaxation instability can develop, and the jet is always unstable for some values of (λ, τ), irrespective of the values of q and E. As the surface charge q increases, the most unstable mode will change from being axisymmetric to sinuous.

At high Reynolds number, when there is a specific downstream direction in which disturbances evolve, there are arguments for considering spatial modes, but at low Reynolds number, a temporal mode analysis should suffice. Yet the elliptic nature of our problem, coupled with the system's preference for long wavelengths raises the question of the influence of boundary conditions at the top and bottom of the jet. As the jet is free and tends in practice to break up into drops before coming into contact with an obstacle, however, it seems unlikely that the precise downstream boundary condition will be critical.

The source of the weak electric repulsion for $m = 1$ is easily visualized, even though the amplification by the Stokes equations is counter-intuitive. The redistribution of charge caused by a lateral displacement of the jet at some value of z is repelled by a similar displacement half a wavelength down the jet in the opposite direction. Indeed, only for long-wave-length axisymmetric modes does the surface charge resist the deformation.

In a real problem, some thought should be given to appropriate conditions at large r, to allow in particular for a return current. A distant boundary would alter the constant term associated with the logarithm as $k \to 0$ and alter the stability regions slightly.

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